M.Sc (Sem-02) Unit-03

Paper - 11

Open mapping theorem:-

Definition:- An operator T on a normed space X into a normed space Y is called open if TA is an open subset of Y whenever A is an open subset of X i.e. T maps open sets into open sets

1. Statement:- Let B and B' be Banach spaces. If T is a continuous linear transformation of B onto B', then T is an open mapping.

Proof:- We are given that the linear transformation

$$T: B \to B'$$

is continuous and onto and we want to show that T is an open map, that is, that $T(G)$ is an open set in B' for every open set G in B.

Let $y \in T(G)$ be arbitrary.

Then $y = T(x)$ for some $x \in G$.

Since G is an open set in B, there exists an open sphere $S(x, r)$ in B centred at x such that $S(x, r) \subset G$.

But as remarked earlier we can write

$S(x, r) = x + S$, where S_r is an open sphere in B centred at origin. Thus

$$x + S_r \subset G \quad \text{———} (1)$$

By our lemma, there exists an open sphere S'_ϵ in B' centred at origin such that $S'_\epsilon \subset T(S_r)$.

$$\therefore\ y + S'_\epsilon \subset y + T[S_r] = T(x) + T[S_r] = T[x + S_r]$$

$$\text{or } S'(y, \epsilon) \subset T(x + S_r) \qquad [\because y + S'_\epsilon = S'(y, \epsilon)]$$

$$\subset T(G) \text{ by (1)}$$

Thus we have shown that to each point $y \in T(G)$ there exists an open sphere in B' centred at y and contained in $T(G)$ and consequently $T[G]$ is an open set. ~~thus~~

Proved.

2. Theorem:- Let B and B' be Banach spaces and let T be a one-one continuous linear transformation

of B onto B'. Then T is a homomorphism. In particular, T^{-1} is automatically continuous.

Proof — Since T is one-one onto and continuous, we need only prove that T is an open mapping. So the proof of this theorem is exactly the same as of the previous theorem.

Note that the hypothesis of theorem 1 is in sufficient to infer that T is a homomorphism in as much as T is not given to be a one-one mapping there. It is this additional property which makes T a homomorphism.

Theorem 2 is often known as Banach's Theorem.

Theorem: Let P be a projection on a Banach space B and let M and N be its range and null space respectively. Then M and N are closed linear manifolds (subspaces) of B such that $B = M \oplus N$

Proof — P is a projection on a Banach space

$\Leftrightarrow$ (i) P is a projection in the algebraic sense and

(ii) P is continuous.

But (i) implies that $B = M \oplus N$ where M and N are range and null spaces of P respectively. Now we use (ii) to prove that M and N are closed subspaces of B.

By definition of null space,

$$N = \{x : P(x) = 0\} = P^{-1}(\{0\}).$$

Since P is continuous and $\{0\}$ is closed in B, it follows that $P^{-1}\{0\}$ is a closed subspace of B.

Again, $M = \{x : P(x) = x\} = \{x : (I-P)(x) = 0\}$

i.e. M is the null space of I-P. But, since the identity map I is always continuous, I-P is also continuous. Hence, as shown above, its null space M must be closed.

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